

On the kinetic theory of vehicular traffic flow: Chapman-Enskog expansion versus Grad's moment method

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Abstract

Based on a Boltzmann-like traffic equation for aggressive drivers we construct in this paper a second-order continuum traffic model which is similar to the Navier-Stokes equations for viscous fluids by applying two well-known methods of gas-kinetic theory, namely: the Chapman-Enskog method and the method of moments of Grad. The viscosity coefficient appearing in our macroscopic traffic model is not introduced in an ad hoc way - as in other second-order traffic flow models - but comes into play through the derivation of a first-order constitutive relation for the traffic pressure. Numerical simulations show that our Navier-Stokes-like traffic model satisfies the anisotropy condition and produces numerical results which are consistent with our daily experiences in real traffic.

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I. INTRODUCTION

Because of the great variety of phenomena present in the motion of vehicles along highways or in urban roads, traffic dynamics has attracted the attention of a large number of researchers in the last decades. Traditionally, three different approaches can be used to study traffic flow problems, namely: a purely microscopic approach in which the acceleration of a driver-vehicle unit is determined by other vehicles moving in traffic flow, a macroscopic approach which describes the collective motion of the vehicles as the one-dimensional compressible flow of a fluid, and a mesoscopic approach which specifies the individual behavior of the vehicles by means of probability distribution functions.

A reading of relevant literature shows that since 1955, when Lighthill and Whitham [1] proposed the first continuum model to describe traffic flow, much progress has been made in the development of macroscopic (fluid-type) models on the one hand, and of microscopic (follow-the-leader) models on the other hand. The first mesoscopic (or gas-kinetic) traffic flow model just appeared in 1960, when Prigogine and Andrews [2] wrote a Boltzmann-like equation to describe the time evolution of a one-vehicle distribution function in a phase-space where the position and the velocity of the vehicles plays a role. Until the 1990s, mesoscopic traffic models do not get much attention from scientists due to their lack of ability to describe traffic operations outside of the free-flow regime. Additionally, compared to macroscopic traffic flow models, gas-kinetic traffic models have a large number of independent variables that increase the computational complexity. However, in the last decade, the scientific community's interest by mesoscopic traffic models resurrected with the publication of some works that apply these models to derive macroscopic traffic models (see, for example, the papers of Helbing [3], Hoogendoorn and Bovy [4], Wagner et al. [5]). Macroscopic equations for relevant traffic variables can be derived from a Boltzmann-like traffic equation by averaging over the instantaneous velocity of the vehicles. This is a well-known procedure in the kinetic theory, nevertheless its application leads to a closure problem, i.e., there are some quantities which must be evaluated with constitutive relations in order to obtain a system of closed equations. The analogy with well-established methods of the kinetic theory of gases - such as the Chapman-Enskog method [6] or the method of moments of Grad [7] - gives us a clue to proceed, provided we have at least a local equilibrium distribution function.

By assuming that motorists drive aggressively [8] we apply both the Chapman-Enskog method and the method of moments of Grad to construct a second-order continuum traffic model which is similar to the Navier-Stokes model for viscous fluids. In the Chapman-Enskog method, constitutive relations

are constructed at successive levels of approximation by expanding the distribution function in power of the mean free path, while in Grad's moment method the distribution function is approximated by an expansion in orthonormal polynomials where the coefficients are related to the moments of the distribution function. The dependence of our constitutive relation for the traffic pressure with the velocity gradient in non-equilibrium situations, drive us to define a traffic viscosity coefficient which, in our case, depends on the traffic state through the density and the mean velocity of the vehicles. As several second-order continuum traffic models, there exists in our Navier-Stokes-like traffic model a characteristic speed that is greater than the average flow velocity. The existence of this faster characteristic speed means that the motion of the vehicles will be influenced by the traffic conditions behind them. This seems to be a drawback of our second-order model since one fundamental principle of traffic flow is that vehicles are anisotropic and respond only to frontal stimuli. However, as pointed out by Helbing and Johansson [9] and J. Yi et al. [10], this faster characteristic speed does not represent a theoretical inconsistency to our Navier-Stokes traffic model since it is related to an eigenmode that decays quickly and, therefore, it cannot emerge by itself. Besides, we check the anisotropic behavior of our second-order continuum traffic model by performing the simulations of a traffic situation where a discontinuity is present, namely: the removal of a blockade scenario.

The organization of the paper is as follows: in Section 2 we present our kinetic traffic model for aggressive drivers. In Section 3 we construct a second-order continuum traffic model - which is similar to the Navier-Stokes equations for viscous fluids - by applying both the Chapman-Enskog method and the method of moments of Grad. In Section 4 we present the results of our numerical simulation. Finally, we give in Section 5 a summary.

II. KINETIC TRAFFIC MODEL

In a mesoscopic description the state of a vehicle at a given instant of time is represented by points in phase space, where the coordinates are the position x and the velocity c of the vehicle. Thus, in analogy with the kinetic theory of gases, an one-vehicle distribution function $f(x, c, t)$ can be defined in such a way that $f(x, c, t) dx dc$ gives at time t the number of vehicles in the road interval between x and $x + dx$ and in the velocity interval between c and $c + dc$. For an uni-directional single-lane road without entrances and exits, the one-vehicle distribution function satisfies the kinetic traffic

equation [11]

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} + \frac{\partial}{\partial c} \left(f \frac{dc}{dt} \right) = \mathcal{Q}(f, f), \quad (1)$$

where the interaction term

$$\mathcal{Q}(f, f) = \int_c^\infty (1 - \mathbf{p})(c' - c) f(x, c, t) f(x, c', t) dc' - \int_0^c (1 - \mathbf{p})(c - c') f(x, c, t) f(x, c', t) dc' \quad (2)$$

describes the deceleration processes due to slower vehicles which cannot be immediately overtaken. The first part of the interaction term corresponds to situations where a vehicle with velocity c' must decelerate to velocity c causing an increase of the one-vehicle distribution function, while the second one describes the decrease of the one-vehicle distribution function due to situations in which vehicles with velocity c must decelerate to even slower velocity c' . The derivation of the interaction term is based on the following hypotheses: (i) vehicles are regarded as point-like objects, (ii) the slowing down process has the probability $(1 - \mathbf{p})$, where \mathbf{p} is the probability of passing, (iii) the velocity of the slow vehicle is not affected by interactions or by being passed, (iv) there is no braking time, (v) only two-vehicle interactions are considered and (vi) vehicular chaos is assumed, in such way that the two-vehicle distribution function can be factorized. The individual acceleration term appearing on the left-hand side of the kinetic traffic equation can be modeled by assuming that vehicles moving with velocity c accelerate exponentially to their desired velocity $c_0 = c_0(x, c, t)$ with a relaxation time τ , i.e.

$$\frac{dc}{dt} = \frac{c_0 - c}{\tau}. \quad (3)$$

In fact, the desired velocity of the vehicles is determined by the average balance among several traffic parameters like legal traffic regulations, weather conditions, road conditions and drivers personality, i.e., it is a phenomenological function. Despite of the variety of traffic parameters that determines the desired velocity of the vehicles, we shall consider in this work the simple relation [8]

$$c_0 = wc, \quad (4)$$

where $w = w(x, t) > 1$ is a positive parameter that depends on traffic conditions along the road. On driver's level, relation (4) indicates that the desired velocity of the vehicles increases as their velocity increases, which is a common feature of aggressive drivers. By assuming that the aggressiveness parameter w depends on position and time only through the vehicular density, we shall consider the

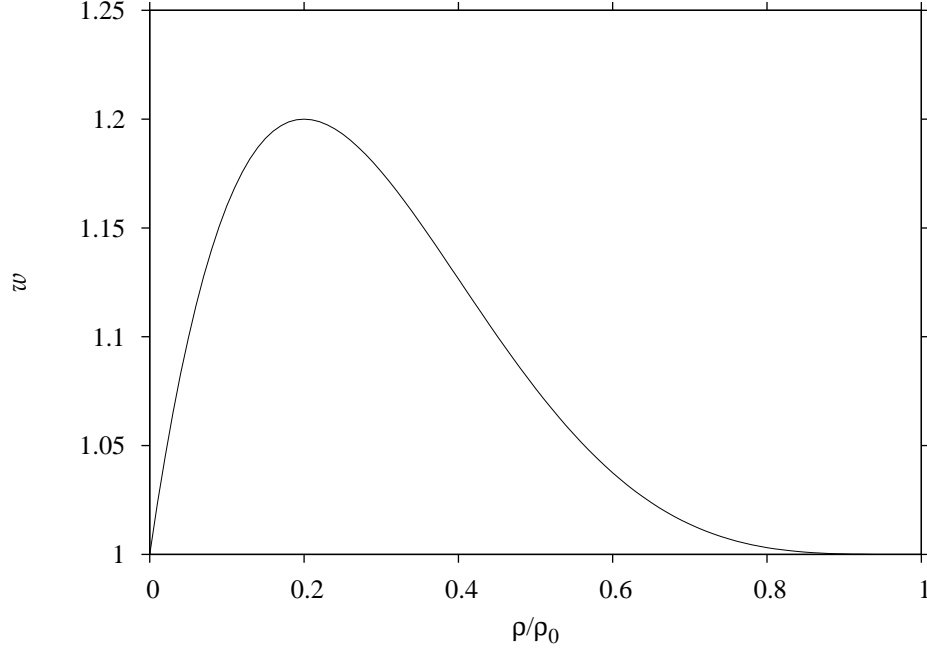


FIG. 1. Dependence of the aggressiveness parameter on the vehicular density

following constitutive relation

$$w = 1 + (w_c - 1) \frac{\rho}{\rho_c} \left(\frac{\rho_0 - \rho}{\rho_0 - \rho_c} \right) \frac{\rho_0 - \rho_c}{\rho_c} \quad (5)$$

where ρ_0 is the jam (or maximum) density, ρ_c is the critical vehicle density (i.e. the density value which maximizes the aggressiveness parameter) and $w_c = w(\rho_c)$ is the maximal value for drivers' aggressiveness. Expression (5) tell us that: (i) when traffic is very dilute the aggressiveness parameter must tend to one, since in this limit vehicles move along the highway at its desired speed, (ii) as the vehicle density increases, the velocity of the vehicles starts to decrease, and drivers try to compensate this situation by increasing their aggressiveness, (iii) when the density exceeds its critical value, it becomes increasingly difficult for the vehicles to move at its desired speed and, in this case, the drivers have no choice except to reduce its aggressiveness, (iv) when the highway is fully occupied, the vehicles can no longer move and the aggressiveness parameter must tend to one. Figure 1 shows the dependence of the aggressiveness parameter on the vehicular density for $\rho_c/\rho_0 = 0.2$ and $w_c = 1.2$. Lastly, it is important to mention that the constitutive relation (5) is not a unique model to the aggressiveness parameter, but just a sound one.

III. SECOND-ORDER CONTINUUM MODEL

The kinetic traffic equation (1) allows the derivation of balance equations for macroscopic traffic variables like the vehicular density

$$\rho(x, t) = \int_0^\infty f(x, c, t) dc \quad (6)$$

and the average velocity

$$v(x, t) = \int_0^\infty c \frac{f(x, c, t)}{\rho(x, t)} dc. \quad (7)$$

The integration of the kinetic traffic equation over all values of the actual velocity of the vehicles yields the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0, \quad (8)$$

while the traffic momentum equation

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x} (\rho v^2 + \varpi) = \rho \frac{w-1}{\tau} v - \rho(1-p)\varpi \quad (9)$$

follows through the multiplication of the kinetic traffic equation with c and the integration over all values of the actual velocity of the vehicles. In the derivation of the traffic momentum equation we have introduced the traffic pressure

$$\varpi(x, t) = \int_0^\infty (c - v)^2 f(x, c, t) dc, \quad (10)$$

and used relations (3) and (4). At this point it is important to emphasize that the balance equations (8) and (9) can only be obtained if the one-vehicle distribution function satisfies the following boundary conditions:

$$\lim_{c \rightarrow 0} f(x, c, t) = 0 \quad \text{and} \quad \lim_{c \rightarrow \infty} f(x, c, t) = 0. \quad (11)$$

Based on the continuity and momentum equations, we can construct a second-order continuum traffic model by specifying the traffic pressure in terms of the vehicular density, the average velocity and their spatial gradients. Since there are a variety of possible constitutive relations which can be borrowed from fluid dynamics, we shall restrict ourselves here to the derivation of a constitutive relation for the traffic pressure which is similar to the usual Navier-Stokes relation for ordinary viscous fluids, i.e., a constitutive relation written in terms of the density, the average velocity and

their first-order spatial gradients. One can achieve this goal by applying, for example, the Chapman-Enskog method or the method of moments of Grad, as they are developed in the kinetic theory of gases. In the Chapman-Enskog method, constitutive relations are constructed at successive levels of approximation by expanding the distribution function in power of the mean free path, while in Grad's moment method the gas-kinetic equation is replaced by a set of balance equations for the moments of the distribution function. To close this set of equations, the distribution function is approximated by an expansion in orthonormal polynomials, where the coefficients are related to the moments of the distribution function. Then, by applying an iteration procedure in the resulting system of field equations, it is possible to derive first-order constitutive relations. Here, we shall apply both methods to derive a first-order constitutive relation for the traffic pressure which is similar to the Navier-Stokes relation for viscous fluids.

A. Chapman-Enskog Method

The basic idea of the Chapman-Enskog method is to expand the distribution function into a power series in the rarefaction parameter ε as

$$f = f^{(0)} + \frac{f^{(1)}}{\varepsilon} + \frac{f^{(2)}}{\varepsilon^2} + \dots = \sum_{r=0}^{\infty} \frac{f^{(r)}}{\varepsilon^r} \quad (12)$$

where $f^{(r)}$ represents successive approximations to the distribution function. In traffic flow problems, the rarefaction parameter is a dimensionless number defined as

$$\varepsilon = \frac{L}{\lambda} \quad (13)$$

where L is the road length and λ denotes the mean distance traveled by a vehicle between successive interactions. Based on the value of the rarefaction parameter, it is possible to identify two traffic flow regimes: (i) the rarefied regime, when $\varepsilon \ll 1$, and (ii) the continuum regime, when $\varepsilon \gg 1$. In the rarefied regime, the processes of acceleration and deceleration of the vehicles can be neglected and the general solution of the kinetic traffic equation in this regime reads

$$f(x, c, t) = f(x - ct, c, t = 0). \quad (14)$$

Otherwise, in the continuum regime, the processes of acceleration and deceleration of the vehicles can not be neglected, so that the distribution function is determined by both the individual acceleration

law and the interaction term. To determine the distribution function in this limit, it is appropriate to write the kinetic traffic equation (1) in the following dimensionless form

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} + \varepsilon \frac{\partial}{\partial c} \left(f \frac{dc}{dt} \right) = \varepsilon \int_0^\infty (1 - \mathbf{p}) (c' - c) f(x, c, t) f(x, c', t) dc' \quad (15)$$

where the position is given in units of the road length, the velocity is given in units of the average velocity, the time is given units of the mean flow time (the mean time required for vehicles to traverse the road) and the vehicular acceleration is given in units of v^2/λ .

Integral equations for the distribution function approximations can be easily obtained if we insert the expansion (12) into the dimensionless kinetic traffic equation (15) and equate equal power of the scale parameter. Hence, we obtain

$$\frac{\partial}{\partial c} \left(f^{(0)} \frac{dc}{dt} \right) = \mathcal{Q}(f^{(0)}, f^{(0)}) \quad (16)$$

and

$$\frac{\partial f^{(r-1)}}{\partial t} + c \frac{\partial f^{(r-1)}}{\partial x} + \frac{\partial}{\partial c} \left(f^{(r)} \frac{dc}{dt} \right) = \mathcal{Q}(f^{(0)}, f^{(r)}) \quad (r \geq 1) \quad (17)$$

where

$$\mathcal{Q}(f^{(0)}, f^{(r)}) = \sum_{n=0}^r \int_0^\infty (1 - \mathbf{p}) (c' - c) f^{(n)} f'^{(r-n)} dc'. \quad (18)$$

By assuming that the vehicular density and the mean velocity are only determined by the zeroth-order approximation of the distribution function, the solution of the integral equation (16) with the individual acceleration law (3) leads to

$$f^{(0)} = \frac{\alpha}{\Gamma(\alpha)} \frac{\rho}{v} \left(\frac{\alpha c}{v} \right)^{\alpha-1} \exp \left(-\frac{\alpha c}{v} \right) \quad (19)$$

where

$$\alpha = \frac{\rho (1 - \mathbf{p}) v \tau}{w - 1}. \quad (20)$$

Expression (19) for the zeroth-order distribution function tell us that the velocity of the vehicles is gamma-distributed with a shape parameter α and a rate parameter $\beta = v/\alpha$ when drivers drive aggressively on the highway. To gain an insight into the shape parameter, let us calculate the velocity variance (or velocity dispersion) in the zeroth-order approximation, i.e.,

$$\Theta = \frac{\int_0^\infty (c - v)^2 f^{(0)} dc}{\int_0^\infty f^{(0)} dc} = \frac{v^2}{\alpha}. \quad (21)$$

From the above expression we verify that the velocity variance depends quadratically on the mean velocity, a fact which can be used to identify the inverse of the shape parameter as the prefactor of the velocity variance. The experimental data reported by Shvetsov and Helbing [12] demonstrate that the prefactor of the velocity variance is almost constant at the low-density region, otherwise it can be taken as a function of the vehicular density. In this work, we shall take this prefactor or, rather, the shape parameter as a constant whose value is greater than unity, so that our gas-kinetic-like traffic model is restricted to low densities.

In order to determine the first-order approximation to the one-vehicle distribution function we introduce expression (19) into the integral equation

$$\frac{\partial f^{(0)}}{\partial t} + c \frac{\partial f^{(0)}}{\partial x} + \frac{\partial}{\partial c} \left(f^{(1)} \frac{dc}{dt} \right) = \mathcal{Q}(f^{(0)}, f^{(1)}) \quad (22)$$

which follows from (17) by taking $r = 1$. By utilizing the constraints

$$\int_0^\infty f^{(r)} dc = 0 \quad \text{and} \quad \int_0^\infty c f^{(r)} dc = 0 \quad (23)$$

which are valid for $r \geq 1$ we obtain

$$f^{(0)} \left[\frac{\dot{\rho}}{\rho} + \frac{v}{\rho} \left(\frac{c}{v} - 1 \right) \frac{\partial \rho}{\partial x} + \alpha \left(\frac{c}{v} - 1 \right) \frac{\dot{v}}{v} + \alpha \left(\frac{c}{v} - 1 \right)^2 \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial c} \left(f^{(1)} \frac{dc}{dt} \right) = -\rho(1 - \mathbf{p})(c - v)f^{(1)} \quad (24)$$

where the dot denotes the material time derivative. Integration of equation (24) over all values of the actual velocity leads to the continuity equation

$$\frac{\dot{\rho}}{\rho} = -\frac{\partial v}{\partial x}. \quad (25)$$

Besides, if we multiply equation (24) with c and integrate over all values of the actual velocity we get

$$\alpha \frac{\dot{v}}{v} = -\frac{v}{\rho} \frac{\partial \rho}{\partial x} - 2 \frac{\partial v}{\partial x} - \alpha(1 - \mathbf{p})v \int_0^\infty \left(\frac{c'}{v} - 1 \right)^2 f'^{(1)} dc'. \quad (26)$$

If equations (25) and (26) are used to eliminate the material time derivatives appearing in equation (24) we obtain

$$\begin{aligned} f^{(0)} \left[\alpha \left(\frac{c}{v} - 1 \right)^2 - 2 \left(\frac{c}{v} - 1 \right) - 1 \right] \frac{\partial v}{\partial x} + \frac{\partial}{\partial c} \left(f^{(1)} \frac{w-1}{\tau} c \right) + \alpha \left(\frac{c}{v} - 1 \right) \frac{w-1}{\tau} f^{(1)} \\ = f^{(0)} \alpha (1 - \mathbf{p}) v \left(\frac{c}{v} - 1 \right) \int_0^\infty \left(\frac{c'}{v} - 1 \right)^2 f'^{(1)} dc'. \end{aligned} \quad (27)$$

Based on equation (27) we shall write the first-order approximation to one-vehicle distribution function as

$$f^{(1)} = f^{(0)} \left[a_0 + a_1 \left(\frac{c}{v} - 1 \right) + a_2 \left(\frac{c}{v} - 1 \right)^2 \right] \frac{\partial v}{\partial x} \quad (28)$$

where a_0 , a_1 and a_2 are position and time-dependent coefficients to be determined. Insertion of expression (28) into constraints (23) and equation (27) leads to

$$a_0 = \frac{a_1}{2} = -\frac{a_2}{\alpha} = \frac{\tau}{2(w-1)}. \quad (29)$$

Note that the dependence of the expansion coefficients with position and time comes through the parameter w that measures the average aggressiveness of drivers. Hence, the first-order approximation to the distribution function reads

$$f^{(1)} = -\frac{f^{(0)}}{2} \frac{\tau}{(w-1)} \left[\alpha \left(\frac{c}{v} - 1 \right)^2 - 2 \left(\frac{c}{v} - 1 \right) - 1 \right] \frac{\partial v}{\partial x}. \quad (30)$$

Once the one-vehicle distribution function is a known function of the basic traffic variables and their first-order spatial gradients, the traffic pressure can be directly calculated as

$$\varpi = \int_0^\infty (c-v)^2 (f^{(0)} + f^{(1)}) dc = \frac{\rho v^2}{\alpha} - 2 \frac{\rho v^2}{\alpha} \tau_0 \left(\frac{\alpha+1}{\alpha} \right) \frac{\partial v}{\partial x} \quad (31)$$

where

$$\tau_0 = \frac{\tau}{2(w-1)}. \quad (32)$$

The constitutive relation (31) for the traffic pressure has a similar form to the Navier-Stokes relation for viscous fluids since in non-equilibrium situations both constitutive relations depend on the velocity gradient. Based on this similarity, it is possible to define a traffic viscosity coefficient

$$\mu = \mu(\rho, v) = 2 \frac{\rho v^2}{\alpha} \tau_0 \left(\frac{\alpha+1}{\alpha} \right) \quad (33)$$

which depends on the traffic state through the vehicular density and the average velocity. At this point, it is important to mention that a similar constitutive relation for the traffic pressure was derived by Velasco and Marques Jr. [8] by applying a simplified version of the Chapman-Enskog method to the reduced Pavani-Fontana traffic equation [13]. In their formalism, the collective relaxation time τ_0 appears as a free adjustable parameter of order of the mean vehicular interaction time, and it was introduced by means of a relaxation approximation performed in the interaction term.

B. Grad's Moment Method

In Grad's moment method a macroscopic description of traffic flow is based on macroscopic traffic variables like the vehicular density, the average velocity and the central moments of the distribution function

$$m_k(x, t) = \int_0^\infty (c - v)^k f(x, c, t) dc \quad (k \geq 2). \quad (34)$$

The balance equations governing the dynamical behavior of these macroscopic traffic variables are the continuity equation (8), the traffic momentum equation (9) and the balance equations

$$\begin{aligned} \frac{\partial m_k}{\partial t} + \frac{\partial}{\partial x} (m_k v + m_{k+1}) + k m_k \frac{\partial v}{\partial x} - k \frac{m_{k-1}}{\rho} \frac{\partial \varpi}{\partial x} - k \frac{w-1}{\tau} m_k \\ = -\rho(1-p) \left(m_{k+1} - k \frac{m_{k-1}}{\rho} \varpi \right). \end{aligned} \quad (35)$$

In deriving the balance equations (35) for the central moments we have multiplied the kinetic traffic equation (1) by $(c - v)^k$, and subsequently integrated with respect to the actual velocity of the vehicles.

Clearly, we can see that the balance equations (8), (9) and (35) form a non-closed system of field equations for the determination of the moments ρ , v and m_k , since the balance equation for the central moment m_k contains the central moment m_{k+1} which is not a priori related to the lower order moments. The dependence of the central moment m_{k+1} upon the moments ρ , v and m_k can be attained if we know the distribution function as a function of ρ , v and m_k . In the method of the moments such normal solution is found by an expansion around a zeroth-order distribution function (19), i.e., we write the distribution function as

$$f(x, c, t) = f^{(0)}(x, c, t) \sum_{n=0}^{\infty} C_n(x, t) P_n(c) \quad (36)$$

where $C_n(x, t)$ are position and time-dependent expansion coefficients and $P_n(c)$ are orthonormal polynomials in the actual velocity of the vehicles. Since in the zeroth-order approximation the velocity of the vehicles is gamma-distributed, it is possible to construct the orthonormal polynomials $P_n(c)$ by applying the condition

$$\int_0^\infty \Phi(s) P_n(s) P_m(s) ds = \delta_{nm} \quad (37)$$

where $s = \alpha c/v$ is the dimensionless instantaneous velocity and

$$\Phi(s) = \frac{s^{\alpha-1}e^{-s}}{\Gamma(\alpha)} \quad (38)$$

is the probability density function of the gamma distribution. From the orthonormality condition (37) we verify that the first polynomials read

$$P_0(s) = 1, \quad (39)$$

$$P_1(s) = \frac{s - \alpha}{\sqrt{\alpha}}, \quad (40)$$

$$P_2(s) = \frac{s^2 - 2(\alpha + 1)s + \alpha(\alpha + 1)}{\sqrt{2\alpha(\alpha + 1)}}, \quad (41)$$

$$P_3(s) = \frac{s^3 - 3(\alpha + 2)s^2 + 3(\alpha + 1)(\alpha + 2)s - \alpha(\alpha + 1)(\alpha + 2)}{\sqrt{6\alpha(\alpha + 1)(\alpha + 2)}}. \quad (42)$$

We can easily verify from the above expressions that the orthonormal polynomials $P_n(s)$ are related to the associated Laguerre polynomials (see the textbook of Arfken [14]) and they are given by the formula

$$P_n(s) = \frac{(-1)^n}{s^{\alpha-1}e^{-s}} \sqrt{\frac{\Gamma(\alpha)}{n! \Gamma(\alpha + n)}} \frac{d^n}{ds^n} (s^{n+\alpha-1}e^{-s}). \quad (43)$$

By using the orthonormality condition (37) the position and time-dependent expansion coefficients C_n can be determined as follows:

$$\int_0^\infty P_n(c) f(x, c, t) dc = \rho \sum_{m=0}^\infty C_m \int_0^\infty \Phi(s) P_n(s) P_m(s) ds = \rho C_n. \quad (44)$$

Thus, we conclude that the expansion coefficients C_n are related directly to the moments of the distribution function. For example, the first coefficients read

$$C_0 = 1, \quad (45)$$

$$C_1 = 0, \quad (46)$$

$$C_2 = \sqrt{\frac{\alpha}{2(\alpha + 1)}} \frac{\varpi - \varpi_0}{\varpi_0}, \quad (47)$$

$$C_3 = \sqrt{\frac{2\alpha}{3(\alpha + 1)(\alpha + 2)}} \left(\frac{\phi - \phi_0}{\phi_0} - 3 \frac{\varpi - \varpi_0}{\varpi_0} \right), \quad (48)$$

where

$$\phi(x, t) = m_3(x, t) = \int_0^\infty (c - v)^3 f(x, c, t) dc \quad (49)$$

is the third-order central moment. Besides,

$$\varpi_0(x, t) = \int_0^\infty (c - v)^2 f^{(0)}(x, c, t) dc = \frac{\rho v^2}{\alpha} \quad (50)$$

and

$$\phi_0(x, t) = \int_0^\infty (c - v)^3 f^{(0)}(x, c, t) dc = 2 \frac{\rho v^3}{\alpha^2} \quad (51)$$

are the values of the second and third-order central moments in the zeroth-order approximation. Insertion of the position and time-dependent coefficients into the expansion of the distribution function allows us to write it in terms of the velocity polynomials and the macroscopic traffic variables. Note that each coefficient in the expansion of the distribution function introduces a new macroscopic traffic variable, so that it is possible to choose which relevant variables we want to use in our macroscopic traffic description.

Let us now construct a continuum traffic flow model based only on three traffic variables, namely: the vehicular density, the average velocity and the traffic pressure. In this case, the balance equations governing the dynamical behavior of traffic variables are the continuity equation (8), the momentum equation (9) and traffic pressure equation

$$\frac{\partial \varpi}{\partial t} + \frac{\partial}{\partial x} (\varpi v + \phi) + 2 \varpi \frac{\partial v}{\partial x} - 2 \frac{w - 1}{\tau} \varpi = -\rho(1 - \mathbf{p})\phi. \quad (52)$$

The balance equations (8), (9) and (52) becomes a system of field equations for the determination of ρ , v and ϖ , if a relationship can be established between these variables and the third-order central moment ϕ . In order to achieve this goal, expansion (36) for the distribution function is taken with $C_n(x, t) = 0$ for $n \geq 3$, so that we have

$$f = f^{(0)} \left\{ 1 + \frac{s^2 - 2(\alpha + 1)s + \alpha(\alpha + 1)}{2(\alpha + 1)} \frac{\varpi - \varpi_0}{\varpi_0} \right\}. \quad (53)$$

A comparison of the distribution function for the first order Chapmann-Enskog expansion (30) and the one obtained by Grad's method (53) shows that the last expression (i) involves an extended set of variables, note that in this formalism the traffic pressure is an independent variable, and (ii) depends only on the macroscopic variables not on the gradients. If the traffic pressure is calculated assuming (53), we realize that the result is different from (31). To reconcile these differences we will carry

out an standard iterative procedure known in the kinetic theory of gases as the Maxwellian iteration procedure.

Insertion of the distribution function (53) into expression (49) leads, after a simple integration, to the following constitutive relation for the third-order central moment:

$$\phi = 3 \frac{\phi_0}{\varpi_0} \left(\varpi - \frac{2}{3} \varpi_0 \right). \quad (54)$$

If we introduce the constitutive relation (54) into the balance equations (8), (9) and (52) we obtain a system of field equations for ρ , v and ϖ or, equivalently, for ρ , v and $\widehat{\varpi}$, where $\widehat{\varpi} = \varpi - \varpi_0$ is the so-called traffic pressure deviator. Hence, after some algebra, we get

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0, \quad (55)$$

$$\rho \frac{\partial v}{\partial t} + \frac{\varpi_0}{\rho} \frac{\partial \rho}{\partial x} + (\alpha + 2) \frac{\varpi_0}{v} \frac{\partial v}{\partial x} + \frac{\partial \widehat{\varpi}}{\partial x} = \rho \frac{w - 1}{\tau} v - \rho (1 - \mathbf{p})(\varpi_0 + \widehat{\varpi}), \quad (56)$$

$$\frac{\partial \widehat{\varpi}}{\partial t} + \left(\frac{\alpha + 4}{2} \right) \frac{\phi_0}{\varpi_0} \frac{\partial \widehat{\varpi}}{\partial x} + 3 \left(\frac{\alpha + 2}{\alpha} \right) \widehat{\varpi} \frac{\partial v}{\partial x} + 2 \varpi_0 \left(\frac{\alpha + 1}{\alpha} \right) \frac{\partial v}{\partial x} = - \frac{\widehat{\varpi}}{\tau_0}. \quad (57)$$

As in the kinetic theory of gases, one can transform the balance equation (57) into an approximate constitutive relation for the traffic pressure deviator by applying a method akin with the Maxwellian iteration procedure [15]. For the first-iteration step we insert, on the left-hand side, the value of the traffic pressure deviator in the zeroth-order approximation, namely $\widehat{\varpi} = 0$, and get, on the right-hand side, the first iterated value

$$\widehat{\varpi} = -2 \frac{\rho v^2}{\alpha} \tau_0 \left(\frac{\alpha + 1}{\alpha} \right) \frac{\partial v}{\partial x}. \quad (58)$$

The above expression for the traffic pressure deviator is identical to that one obtained by the Chapman-Enskog procedure, a fact that allow us to affirm that Chapman-Enskog expansion and Grad's moment method are both physically and mathematically equivalent, at least in first-order approximation.

C. Navier-Stokes-like Traffic Model

Insertion of the constitutive relation (31) for the traffic pressure into the balance equations (8) and (9) leads to the so-called Navier-Stokes-like traffic model which can be written in the following conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U}) \quad (59)$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho v^2 + \rho c^2 \end{pmatrix} \quad \text{and} \quad \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ \rho \frac{u(\rho, v) - v}{\tau} - b v_x + (\mu v_x)_x \end{pmatrix}. \quad (60)$$

In the above expressions we have introduced the traffic sound speed $c(v) = \sqrt{\partial \varpi_0 / \partial \rho}$, the optimal velocity function $u(\rho, v) = wv - \tau(1 - \mathbf{p})\varpi_0$ and the anticipation coefficient

$$b(\rho, v) = - \left(\frac{\alpha - 1}{2} \right) \frac{\partial \varpi_0}{\partial v} < 0. \quad (61)$$

In contrast to others macroscopic traffic models, we see that our optimal velocity function does not depend only on the vehicular density, but also on the average velocity and that such dependence is explicitly determined by the average desired velocity of the vehicles reduced by a term arising from deceleration processes due to vehicle interactions. Besides, in our macroscopic traffic flow model, traffic viscosity is not introduced in an ad hoc way - as in some of the most popular second-order continuum traffic models - but it comes into play via an interaction procedure and reflects the way drivers anticipate traffic situation on the basis of second-order spatial changes in the average velocity.

Finally, we close this section by remarking that the eigenvalues Λ of the conservative flux-Jacobian matrix

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} = \begin{pmatrix} 0 & 1 \\ -\frac{\alpha + 1}{\alpha} v^2 & 2 \frac{\alpha + 1}{\alpha} v \end{pmatrix} \quad (62)$$

determine how traffic information (such as slow-downs and speed-ups) is transmitted in a traffic stream. These eigenvalues, also known as characteristic speeds, are found by setting

$$\det[\mathbf{A}(\mathbf{U}) - \Lambda \mathbf{I}] = 0, \quad (63)$$

where \mathbf{I} is the identity matrix. Hence, our Navier-Stokes-like traffic model has two real and distinct characteristic speeds, namely:

$$\Lambda_{1,2} = \frac{\alpha + 1 \mp \sqrt{\alpha + 1}}{\alpha} v. \quad (64)$$

Note that one of the characteristic speeds is larger than the average traffic flow velocity indicating that traffic disturbances propagate in the downstream direction. This fact has been criticized by Daganzo [16] as a major deficiency of most second-order traffic models. However, as shown by Helbing and Johanson [9] in a recent publication, this fact does not constitute a theoretical inconsistency of our second-order macroscopic traffic model since the perturbation that travels faster than traffic decays quickly.

IV. NUMERICAL EXAMPLE

Since the Navier-Stokes-like traffic equations (59) form a hyperbolic system of partial differential equations which involve smooth as well as discontinuous solutions, it is suitable to use a conservative numerical scheme to solve them accurately. However, as pointed out in the literature, conservative numerical schemes are only appropriate for solving the homogeneous part of the system, i.e. when the source term $\mathbf{S}(\mathbf{U})$ vanishes. In order to deal with the source term, the following fractional-step scheme [17]

$$\mathbf{U}_i^* = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} (\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}) \quad (65)$$

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^* + \frac{\Delta t}{2} (\mathbf{S}(\mathbf{U}_i^n) + \mathbf{S}(\mathbf{U}_i^*)) \quad (66)$$

can be employed, where Δx is the grid size, Δt is the time increment and $\mathbf{F}_{i+1/2} = \mathbf{F}(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$ is the intercell numerical flux. The intercell numerical flux $\mathbf{F}_{i+1/2}$ approximates the time-integral average of flux across the cell interface between cells i and $i+1$, where the i -th cell is given by the spatial interval between $x_{i-1/2}$ and $x_{i+1/2}$. One can find in the literature several classes of methods that provide stable and consistent approximations to the intercell numerical flux. Here, we determine the numerical flux by employing a first order upwind-type scheme based on Roe's flux-difference splitting [18].

The application of Roe's scheme to the homogeneous version of our macroscopic traffic model yields a conservative method whose numerical flux function is computed as

$$\mathbf{F}_{i+1/2} = \frac{\mathbf{F}(\mathbf{U}_i) + \mathbf{F}(\mathbf{U}_{i+1})}{2} - \frac{1}{2} \sum_k \sigma_k |\Lambda_k| \mathbf{e}_k, \quad (67)$$

where Λ_k , σ_k and \mathbf{e}_k are respectively the eigenvalues, the wave strengths and the right-eigenvectors of the flux-Jacobian matrix $\mathbf{A}(\bar{\mathbf{U}})$ at Roe's average state $\bar{\mathbf{U}}$. For the Navier-Stokes-like traffic model these quantities are expressed as:

$$\Lambda_{1,2} = \frac{\alpha + 1 \mp \sqrt{\alpha + 1}}{\alpha} \bar{v}, \quad (68)$$

$$\sigma_{1,2} = \mp \frac{\alpha}{2\sqrt{\alpha + 1}} \left[\frac{\Delta(\rho v)}{\bar{v}} - \frac{\alpha + 1 \pm \sqrt{\alpha + 1}}{\alpha} \Delta\rho \right], \quad (69)$$

$$\mathbf{e}_{1,2} = \begin{pmatrix} 1 \\ \frac{\alpha + 1 \mp \sqrt{\alpha + 1}}{\alpha} \bar{v} \end{pmatrix}, \quad (70)$$

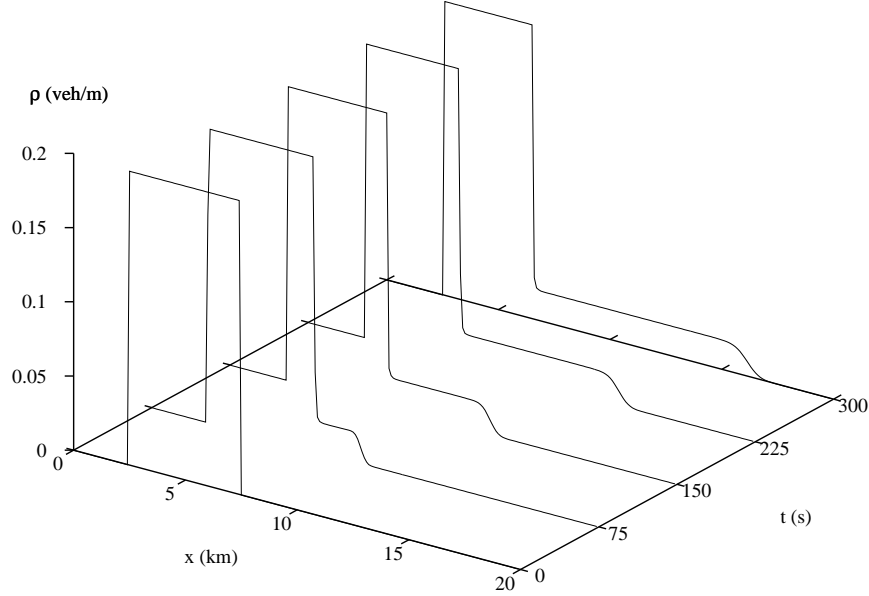


FIG. 2. Time evolution of the vehicle density for the removal of blockade scenario

where $\Delta \mathbf{U} = \mathbf{U}_{i+1} - \mathbf{U}_i$. As usual, Roe's average state is determined by imposing the shock-capturing property

$$\Delta \mathbf{F} = \mathbf{F}(\mathbf{U}_{i+1}) - \mathbf{F}(\mathbf{U}_i) = \mathbf{A}(\bar{\mathbf{U}}) \Delta \mathbf{U}. \quad (71)$$

From (71) we verify that the first equation is identically satisfied, while the second one gives the following average for the velocity:

$$\bar{v} = \frac{\sqrt{\rho_i} v_i + \sqrt{\rho_{i+1}} v_{i+1}}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}}. \quad (72)$$

Note that no representation for average traffic density is required in our Navier-Stokes-like traffic model, since Roe's average (72) for the flow velocity completely defines the flux-Jacobian matrix. Concerning now the discretization of the source term we take

$$\mathbf{S}(\mathbf{U}_i) = \rho_i \frac{u(\rho_i, v_i) - v_i}{\tau} - \frac{b(\rho_i, v_i)}{2} \frac{v_{i+1} - v_{i-1}}{\Delta x} + \frac{\mu_{i+1/2}(v_{i+1} - v_i) - \mu_{i-1/2}(v_i - v_{i-1})}{(\Delta x)^2}, \quad (73)$$

where

$$\mu_{i\pm 1/2} = \frac{\mu(\rho_i, v_i) + \mu(\rho_{i\pm 1}, v_{i\pm 1})}{2}. \quad (74)$$

Completing the description of the numerical scheme it is worth to mention that the numerical stability of explicit difference schemes is ensured by the Courant-Friedrich-Lewy condition

$$\frac{\Delta t}{\Delta x} \max_k |\Lambda_k| \leq 1 \quad (75)$$

which guarantees that information propagates only through a single cell at each time step.

We illustrate now the numerical scheme presented above by considering a traffic situation where a queue of nearly motionless vehicles is present in a certain road region. At the initial time, the blockade at the head of the queue is removed and vehicles flow into the empty part of the roadway. For simulation of this traffic scenario, we consider the following initial conditions on a 20 km circular road:

$$\rho(x, 0) = \begin{cases} \rho_*, & \text{if } 2.5 \text{ km} < x < 7.5 \text{ km} \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad v(x, 0) = v_e(\rho(x, 0)), \quad (76)$$

where ρ_* is the vehicle density in the queue and $v_e(\rho)$ is the equilibrium speed-density function. Here, we assume an equilibrium speed-density relation of the form (see Bando et al. [19])

$$v_e(\rho) = \frac{v_0}{2} (\tanh(\rho_0/\rho - a) + \tanh(a)), \quad (77)$$

where v_0 is the free-flow velocity and a is a positive dimensionless constant. Furthermore, we impose periodic boundary conditions and choose the following values for the model parameters:

$$\alpha = 125, \quad v_0 = 30 \text{ m/s}, \quad \rho_0 = 0.2 \text{ veh/m}, \quad \rho_* = 0.198 \text{ veh/m}, \quad a = 3.9 \quad \text{and} \quad \tau = 8 \text{ s}. \quad (78)$$

Regarding the probability of passing it is usual to assume that this quantity depends only on the vehicular density in a linear way. However, as pointed out by Hoogendoorn and Bovy [20] in their report, an expression for the probability of passing that depends both on the vehicular density and the mean velocity can be obtained if we set $u(\rho, v) = v_e(\rho)$. By equating the optimal velocity to the equilibrium speed-density function we are in fact replacing the individual (microscopic) process of deceleration by some collective (macroscopic) relaxation process to an equilibrium traffic state.

Figures 2 and 3 show that after the removal of the blockade the vehicles at the head of the queue move downstream into the empty region with the free-flow velocity, while vehicles at the tail of the queue remain at their location. Although the traffic conditions upstream are free-flow we observe from these figures that vehicles do not flow backwards into the empty region, a fact that allow us to say that our Navier-Stokes-like traffic model satisfies the anisotropy condition and

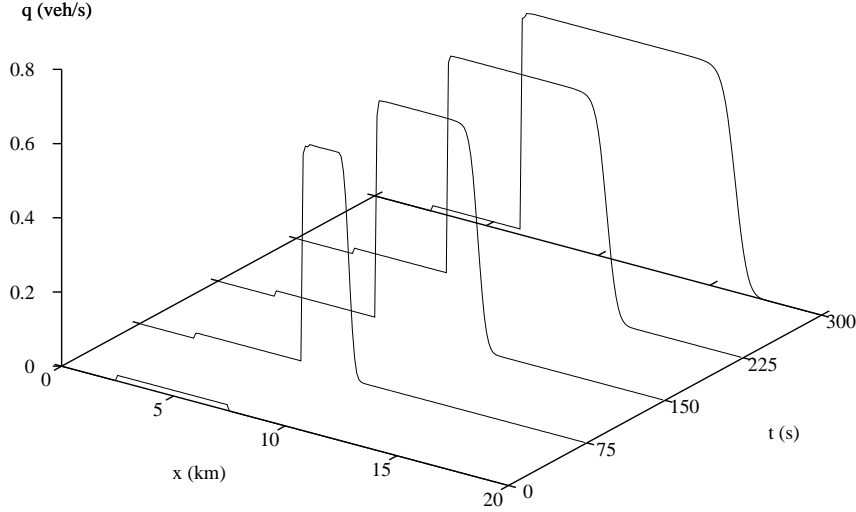


FIG. 3. Time evolution of the traffic flow for the removal of blockade scenario

produces numerical results which are similar to traffic operations in real-life traffic. Similar results were obtained by Hoogendoorn [21] through numerical simulations performed on the single user-class version of his macroscopic multiple user-class traffic flow model. Hoogendoorn's macroscopic multiple user-class traffic model was derived from mesoscopic principles which encompass contributions of drivers acceleration towards their user-class specific desired velocity and contributions resulting from interactions between vehicles of the same and different classes. Besides, the velocity variance is introduced as an additional basic field describing deviations from the average velocity within the user-classes.

V. SUMMARY

By applying both Chapman-Enskog expansion and Grad's moment method we constructed a second-order continuum traffic model which is very similar to the Navier-Stokes model for viscous fluids. The constitutive relation for the traffic pressure obtained by the method of moments of Grad is identical to that one obtained by the Chapman-Enskog procedure, a fact that allow us to affirm that Chapman-Enskog expansion and Grad's moment method are both physically and mathematically equivalent, at least in first-order approximation. Besides, in contrast to others second-order macroscopic

traffic models, our traffic viscosity coefficient - which depends on the traffic state through the vehicle density and the mean velocity - is not introduced in an ad hoc way, but comes into play through the derivation of a constitutive relation to the traffic pressure. Numerical simulation of a traffic scenario where a discontinuity is present show that our macroscopic traffic model satisfies the anisotropy condition and produces numerical results which are similar to traffic operations in real-life traffic.

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